Statistikk, Innlevering 5, Vegard G. Jervell

Oppg. 1a) $Var[X] = \sigma_A^2 = 0{,}2^2$

$Var[Y] = \sigma_B^2 = 0{,}1^2$

En god estimator er forventningsrett og har raskt avtagende og minst mulig varians

$Var[\hat{\mu}] = Var[Y] = 0{,}1^2 = 10^{-2}$

$Var[\tilde{\mu}] = \frac{1}{4}\left(Var[X] + Var[Y]\right) = 1{,}25 \cdot 10^{-2}$

$Var[\mu^*] = \frac{1}{25} Var[X] + \frac{16}{25} Var[Y] = 8 \cdot 10^{-3} = 0{,}8 \cdot 10^{-2}$

Vil foretrekke μ^* fordi den har lavest varians

1b) $X \sim N(\mu, \sigma_A)$, $Y \sim N(\mu, \sigma_B)$

μ^* er en lin. komb. av X og Y

$\rightarrow \mu^* \sim N(\mu, \sigma_\mu)$

$\frac{\mu^* - \mu}{\sigma_*} \sim N(0,1)$, $\sigma_*^2 = Var[\mu^*]$

$P\left(z_{-\frac{\alpha}{2}} < \frac{\mu^* - \mu}{\sigma_*^2} < z_{\frac{\alpha}{2}}\right) = 1 - \alpha = 0{,}9$

$\rightarrow \alpha = 0{,}1 \rightarrow \frac{\alpha}{2} = 0{,}05$

$P\left(\mu^* - \sigma_*^2 z_{\frac{\alpha}{2}} < \mu < \mu^* + \sigma_*^2 z_{\frac{\alpha}{2}}\right) = 0{,}9$

90% K.I.: $(\mu^* - 0{,}147 , \mu^* + 0{,}147)$

Oppg. 2a)

$$P(X > 20) = 1 - P(X \leq 20)$$

$$P(X > 20) = 0{,}083$$

$$P(10 \leq X < 20) = P(X \leq 19) - P(X \leq 10)$$

$$= 0{,}8752 - 0{,}1185$$

$$P(10 \leq X < 20) = 0{,}7567$$

$$f(x;\lambda) = \frac{\lambda^x e^{-\lambda}}{x!} \rightarrow E[X] = \sum_{x=0}^{\infty} x f(x;\lambda) = \lambda$$

$$E[X] = 15$$

2b) $\hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n} X_i$, X_i u.i.f.

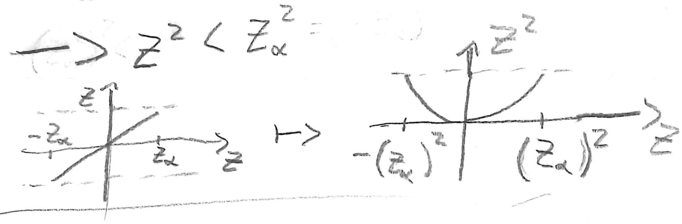
$$\rightarrow \hat{\lambda} \sim N\left(\lambda, \frac{\sigma_x}{\sqrt{n}}\right)$$

$$Var[X_i] = \lambda \rightarrow Var[\hat{\lambda}] = \frac{\lambda}{n}$$

$$\frac{\hat{\lambda} - \lambda}{\sqrt{\frac{\lambda}{n}}} = Z \sim N(0,1)$$

$$-z_\alpha < Z < z_\alpha$$

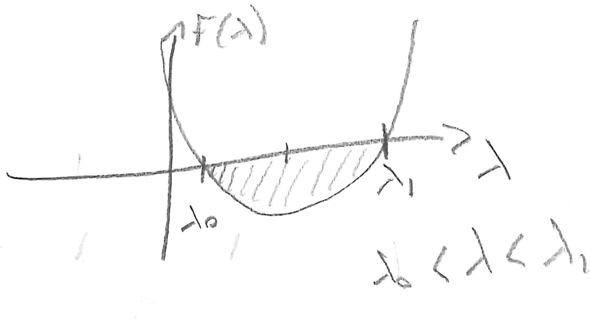
$$\rightarrow Z^2 < z_\alpha^2$$



$$P\left(z_{-\frac{\alpha}{2}} < Z < z_{\frac{\alpha}{2}}\right) = 1 - \alpha$$

$$z_{-\frac{\alpha}{2}} < \sqrt{\frac{n}{\lambda}}(\hat{\lambda} - \lambda)$$

$$\left(z_{-\frac{\alpha}{2}}\right)^2 > \frac{n}{\lambda}(\hat{\lambda} - \lambda)(\hat{\lambda} - \lambda)$$



$$\lambda_0 < \lambda < \lambda_1$$

$$(\hat{\lambda} - \lambda)(\hat{\lambda} - \lambda)n - \lambda\left(z_{-\frac{\alpha}{2}}\right)^2 < 0$$

$$\hat{\lambda}^2 - \lambda\left(2\hat{\lambda} + \frac{1}{n}z^2_{-\frac{\alpha}{2}}\right) + \lambda^2 = 0$$

$$\lambda = \frac{2\hat{\lambda} + \frac{1}{n}z^2_{-\frac{\alpha}{2}} \pm \sqrt{\left(2\hat{\lambda} + \frac{1}{n}z^2_{-\frac{\alpha}{2}}\right)^2 - 4\hat{\lambda}^2}}{2}$$

$\alpha = 0{,}01 \rightarrow \underline{z_{\frac{\alpha}{2}} = 2{,}576}$

$\underline{\hat{\lambda} = \frac{1}{n}\sum x_i = 11{,}967}$

$$\lambda = \frac{2\hat{\lambda} + \frac{1}{n} z_{\frac{\alpha}{2}}^2 \pm \sqrt{(2\hat{\lambda} + \frac{1}{n} z_{\frac{\alpha}{2}}^2)^2 - 4\hat{\lambda}^2}}{2}$$

$\lambda_1 = 10{,}478 \wedge \lambda_0 = 13{,}744$

$P(z_{-\frac{\alpha}{2}} < z < z_{\frac{\alpha}{2}}) = 0{,}99$

$\rightarrow$ K.I.: $(10{,}478, 13{,}744)$

2c) $f(x;\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$

$$L(\lambda; n, m) = \left(\prod_{i=1}^{n} f(x_i;\lambda)\right)\left(\prod_{j=1}^{m} f(y_j; \frac{\lambda}{2})\right)$$

$$L(\lambda; n, m) = \frac{\lambda^{\sum_{i=1}^{n} x_i} e^{-n\lambda}}{\prod_{i=1}^{n}(x_i!)} \cdot \frac{\left(\frac{\lambda}{2}\right)^{\sum_{j=1}^{m} y_j} e^{-m\frac{\lambda}{2}}}{\prod_{j=1}^{m}(y_j!)}$$

$l(\lambda; n, m) = \ln(L)$

$$l(\lambda; n, m) = \sum_{i=1}^{n} x_i \ln(\lambda) - n\lambda + \sum_{j=1}^{m} y_j \ln\left(\frac{\lambda}{2}\right) - \frac{m\lambda}{2} - \sum_{i=1}^{n}(x_i!) - \sum_{j=1}^{m}(y_j!)$$

$$\frac{d\, l(\lambda; n, m)}{d\lambda} = 0$$

$$\frac{1}{\lambda}\sum_{i=1}^{n} x_i - n + \frac{2}{\lambda}\sum_{j=1}^{m} y_j - \frac{m}{2} = 0$$

$$\left[\sum_{i=1}^{n} x_i + 2\sum_{j=1}^{m} y_j\right]\frac{1}{n + \frac{m}{2}} = \lambda$$

$$\underline{\hat{\lambda} = \frac{\sum_{i=1}^{n} x_i + 2\sum_{j=1}^{m} y_j}{n + \frac{m}{2}}}$$

Sjekker maksimum:

$$\frac{d^2 l}{d\lambda^2} = \frac{-1}{\lambda^2}\sum x_i - \frac{2}{\lambda^2}\sum y_i < 0 \rightarrow \text{maksimum}$$

Fordi $x_i \geq 0$, $y_i \geq 0$

og $\lambda^2 \geq 0$ $\square$

Oppg. 3)

$$f_T(t;\lambda) = \begin{cases} \lambda e^{-\lambda t}, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

$Z = 2\lambda T$, skal vise $Z \sim \chi_2^2$

$$P(T < t) = \int_0^t f_T(x)\,dx = \left[-e^{-\lambda x}\right]_0^t = 1 - e^{-\lambda t} = F_T(t)$$

$$P(Z < z) = P(2\lambda T < z) = P\left(T < \frac{z}{2\lambda}\right) = F_T\left(\frac{z}{2\lambda}\right) = F_Z(z)$$

$$F_T\left(\frac{z}{2\lambda}\right) = \int_0^z f_Z(x)\,dx$$

$$f_Z(z) = \frac{d\,F_T\left(\frac{z}{2\lambda}\right)}{dz}$$

$$f_Z(z) = \frac{d}{dz}\left(1 - e^{\frac{-z}{2}}\right)$$

$$f_Z(z) = \frac{1}{2}e^{-\frac{z}{2}} \quad \square$$

Oppg. 3B)

Lager SME for λ:

$$L(\lambda) = \prod_{i=1}^{n} f_T(t_i) = \lambda^n \exp\left(-\lambda \sum_{i=1}^{n} t_i\right)$$

$$l(\lambda) = \ln(L) = n\ln(\lambda) - \lambda \sum_{i=1}^{n} t_i$$

$$\frac{dl}{d\lambda} = 0 \rightarrow \hat{\lambda} = \frac{n}{\sum_{i=1}^{n} t_i}$$

$$\hat{\mu} = \frac{1}{\hat{\lambda}} = \frac{1}{n}\sum t_i$$

$$2n\lambda\hat{\mu} = 2\lambda \sum t_i \sim \chi^2_{2n}$$

$$\frac{d^2 l}{d\lambda^2} = \frac{-n}{\lambda^2} < 0$$

Fordi $2\lambda T \sim \chi^2_2$ og $\sum_{i=1}^{n} Z_i \sim \chi^2_{n\nu}$ Hvis $Z_i \sim \chi^2_\nu$ $\forall i$

$$P\left(\chi^2_{1-\frac{\alpha}{2},20} < 2n\lambda\hat{\mu} < \chi^2_{\frac{\alpha}{2},20}\right) = 1-\alpha$$

$\alpha = 0,1 \rightarrow \chi^2_{\frac{\alpha}{2},20} = 10,851$, $\chi^2_{1-\frac{\alpha}{2},20} = 31,410$

90% K.I., $\hat{\lambda} = \frac{1}{\hat{\mu}} = 1,555 \cdot 10^{-3}$

$$P\left(\hat{\lambda}\frac{10,851}{2n} < \lambda < \hat{\lambda}\frac{31,410}{2n}\right) = 0,9$$

KI: $(8,438 \cdot 10^{-4}, 2,442 \cdot 10^{-3})$

$$\frac{1}{n}\sum T_i \sim \Gamma(\alpha, \beta)$$

$$\alpha\beta = \frac{1}{\hat{\lambda}} \quad , \quad \alpha\beta^2 = \frac{1}{n\hat{\lambda}^2}$$

$$\beta = \frac{1}{n\hat{\lambda}} \quad , \quad \alpha = n$$

$$\frac{1}{n}\sum T_i \sim \Gamma\left(n, \frac{1}{n\hat{\lambda}}\right)$$

$$\beta = 2 \; ; \; \alpha = \frac{\nu}{2} \quad \rightarrow \quad \chi^2_\nu$$

$$\hat{\lambda} = \frac{1}{2n} \quad \nu = 2n$$

$$\hat{\mu} \sim \Gamma\left(n, \frac{1}{n\hat{\lambda}}\right)$$