Statistikk, Innlevering 6, Vegard G. Jervell

Oppg. 1a) $X \sim p(x;\lambda)$

$$f(x) = \frac{\lambda^x}{x!} e^{-\lambda}$$

$$L(x) = \prod_{i=1}^{n} f(x_i) = \frac{\lambda^{\sum_i x_i}}{\sum_i x_i!} e^{-n\lambda}$$

$$l(x) = \ln(L(x)) = \sum_i (x_i) \ln(\lambda) - n\lambda - \ln(\sum_i x_i!)$$

$$\frac{dl}{d\lambda} = \frac{\sum_i x_i}{\lambda} - n = 0 \rightarrow \underline{\hat{\lambda} = \frac{1}{n}\sum_i x_i}$$

$$\frac{d^2 l}{d\lambda^2} = \frac{-\sum_i x_i}{\lambda^2}, \quad x_i \geq 0 \ \forall i \rightarrow \underline{\frac{d^2 l}{d\lambda^2} \leq 0} \quad \square$$

Sentralgrenseteoremet: En sum av u.i.f. S.V. vil nærme seg normalfordelingen

$$E(\hat{\lambda}) = E\left(\frac{1}{n}\sum_i x_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(x_i) = \frac{1}{n} \cdot nE(X) = \lambda \quad \square$$

$$Var[\hat{\lambda}] = Var\left[\frac{1}{n}\sum_i x_i\right] = \frac{1}{n^2} Var\left[\sum_{i=1}^{n} x_i\right]$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} Var[X_i] = \frac{1}{n^2} \cdot n\lambda, \quad \text{Fordi } X_i \sim p(x;\lambda)$$

$$Var[\hat{\lambda}] = \frac{\lambda}{n} \quad \square$$

1b) $H_0: \lambda = 10, \quad H_1: \lambda < 10, \quad \hat{\lambda} \sim N(\lambda, \frac{\lambda}{n})$

$$Z := \frac{\hat{\lambda} - \lambda}{\sqrt{(\frac{\lambda}{n})^2}} = \frac{\hat{\lambda} - \lambda}{\lambda / n} \sim N(0,1)$$

$$P(Z \geq Z_\alpha) = 1 - \alpha \rightarrow P(Z \leq -Z_\alpha) = \alpha$$

$$P\left(\hat{\lambda} \leq \lambda - \frac{\lambda}{n} Z_\alpha\right) = \alpha$$

$\sum x_i = 470, \ n = 50 \rightarrow \hat{\lambda} = 9{,}4$

$n = 50, \lambda = 10, \alpha = 0{,}05 \rightarrow \lambda - \frac{\lambda}{n} Z_\alpha = 9{,}671$

$\rightarrow$ Forkast H_0

Det er færre rosiner

1b) $H_0 : \lambda = 10$, $H_1 : \lambda < 10$

$$\hat{\lambda} \sim N\left(\lambda, \left(\frac{\lambda}{n}\right)\right) \; ; \; Z := \frac{\hat{\lambda} - \lambda}{\sqrt{\frac{\lambda}{n}}} \sim N(0,1)$$

$P(\text{Forkast } H_0 \mid H_0) = \alpha$

Forkast H_0 når $Z \leq p = -z_{0,05}$

$\sum x_i = 470$, $n = 50 \to \underline{Z = -1{,}34 > -1{,}645}$

Ikke Forkast H_0

1c) Vil Finne en n s.a.

$$P(\text{Forkast } H_0 \mid \lambda = \lambda_r) = 0{,}9 \; , \; \lambda_r = 9$$

Forkast H_0 når $Z = \frac{\hat{\lambda} - \lambda_0}{\sqrt{\frac{\lambda_0}{n}}} \leq p = -z_{0,05}$

$P\left(\frac{\hat{\lambda} - \lambda_0}{\sqrt{\frac{\lambda_0}{n}}} \leq p\right) = 0{,}9$, $\hat{\lambda} \sim N\left(\lambda_r, \frac{\lambda_r}{n}\right)$

$P\left(\hat{\lambda} \leq \lambda_0 + \sqrt{\frac{\lambda_0}{n}}\, p\right) = 0{,}9$ $\quad Z_r := \frac{\hat{\lambda} - \lambda_r}{\sqrt{\frac{\lambda_r}{n}}} \sim N(0,1)$

$P\left(Z_r \leq \sqrt{\frac{n}{\lambda_r}}\left[\lambda_0 - \lambda_r + p\sqrt{\frac{\lambda_0}{n}}\right]\right) = 0{,}9$

$P\left(Z_r \geq \sqrt{\frac{n}{\lambda_r}}\left[\lambda_0 - \lambda_r + p\sqrt{\frac{\lambda_0}{n}}\right]\right) = 0{,}1$

$$\to \sqrt{n}\left(\frac{\lambda_0 - \lambda_r}{\sqrt{\lambda_r}}\right) + p\sqrt{\frac{\lambda_0}{\lambda_r}} = z_{0,1} \to \sqrt{n} = \frac{z_{0,1} - p\sqrt{\frac{\lambda_0}{\lambda_r}}}{\frac{\lambda_0 - \lambda_r}{\sqrt{\lambda_r}}}$$

$n = 81{,}87$

Etter 82 baller er $P(\text{Forkast } H_0 \mid \lambda = 9) = 0{,}9$

Oppg 2a) definerer $\theta_i = x_i(1-x_i)$

da blir sannsynlighetstettheten til Y:

$$f(y) = \frac{1}{\sqrt{2\pi}\,\sigma_0} \exp\left(-\frac{(y - a\theta)^2}{2\sigma_0^2}\right)$$

Som gir rimelighetsfunksjonen

$$L(a) = \prod_{i=1}^{n} f(y_i) = \left(\frac{1}{\sqrt{2\pi}\,\sigma_0}\right)^n \exp\left(-\frac{\sum (y_i - a\theta_i)^2}{2^n \sigma_0^{2n}}\right)$$

Skal maksimere $L(a)$ mhp. a, lager hjelpefunksjonen

$$l(a) = \ln(L)$$

$$l(a) = -n\ln(\sqrt{2\pi}\,\sigma_0) - (2\sigma_0^2)^{-n} \sum_{i=1}^{n} (y_i - a\theta_i)^2$$

$$\frac{dl}{da} = 0 \rightarrow \quad -(2\sigma_0^2)^{-n} \sum_{i=1}^{n} -2\theta_i(y_i - a\theta_i) = 0$$

$$\sum y_i\theta_i - a\theta_i^2 = 0$$

$$\hat{a} = \frac{\sum y_i\theta_i}{\sum \theta_i^2} = \frac{\sum x_i(1-x_i)y_i}{\sum (x_i[1-x_i])^2} \quad \square$$

2b) $E[\hat{a}] = E\left[\frac{\sum \theta_i y_i}{\sum \theta_i^2}\right] = E\left[\sum c_i y_i\right], \quad c_i = \frac{\theta_i}{\sum_j \theta_j^2}$

$$E[\hat{a}] = \sum_i c_i E[y_i]$$

$$E[\hat{a}] = \sum_i c_i a\theta_i = a \underbrace{\sum_i \frac{\theta_i}{\sum_j \theta_j^2}\theta_i}_{\sum_i \frac{\theta_i^2}{\sum_j \theta_j^2} = \frac{\sum_i \theta_i^2}{\sum_j \theta_j^2} = 1} = a$$

$\underline{\underline{E[\hat{a}] = a}}$

$$Var[\hat{a}] = Var\left[\sum_i c_i Y_i\right]$$

$$= \sum_i c_i^2 \, Var[Y_i] = \sum_{i=1}^{n} \left(\frac{\theta_i}{\sum_j \theta_j} \right)^2 \sigma_0^2$$

$$Var[\hat{a}] = n\sigma_0^2 \left[\frac{\sum \theta_i^2}{\left(\sum_j \theta_j\right)^2} \right]$$

$\hat{a}$ er en lin. komb. av uavhengige $Y_i \sim N(\mu, \sigma_0^2)$

$\rightarrow$ a er normalfordelt

2c) $H_0 : a = 0$, $H_1 : a \neq 0$

Forkast hvis $\hat{a}$ er liten eller stor

$Var[\hat{a}] := \sigma_a^2$, $\hat{a} \sim N(a, \sigma_a^2) \rightarrow Z = \frac{\hat{a} - a}{\sigma_a} \sim N(0,1)$

$$P(Z \leq -z_{\frac{\alpha}{2}} \mid H_0) + P(Z \geq z_{\frac{\alpha}{2}} \mid H_0) = \alpha$$

$\alpha = 0{,}05 \rightarrow z_{\frac{\alpha}{2}} = 1{,}960$

$\sigma_a^2 = 0{,}0169$, $\hat{a} = -0{,}284 \rightarrow Z = -2{,}18 < -z_{\frac{\alpha}{2}}$

$\rightarrow$ Forkast H_0 (ikke ideel blanding)