Matte 4, Øving 7. Vegard G. Jervell

Oppg. 1)

$u_t = c^2 u_{xx}$, $u(x,0) = f(x)$

krav til u og f:

$\lim_{x\to\pm\infty} u(x,t) = 0$, $\lim_{x\to\pm\infty} f(x) = 0$

$$F(u_t) = F(c^2 u_{xx})$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial}{\partial t} u(x,t) e^{-iwx} dx = -c^2 w^2 \hat{u}(w,t)$$

$$\frac{\partial}{\partial t} F(u) = -c^2 w^2 \hat{u}(w,t)$$

$$\hat{u}_t(w,t) = -c^2 w^2 \hat{u}(w,t) \to \hat{u}(w,t) = A e^{-c^2 w^2 t}$$

$$u(x,0) = f(x) \to \hat{u}(w,0) = \hat{f}(w)$$

$$\hat{u}(w,0) = A \to A = \hat{f}(w)$$

$$\hat{u}(w,t) = \hat{f}(w) e^{-c^2 w^2 t}$$

(Vil finne $g(x) \mid F(g) = e^{-c^2 w^2 t}$)

$$F(e^{-ax^2}) = \frac{1}{\sqrt{2a}} e^{-\frac{w^2}{4a}}$$

$$\frac{1}{4a} = c^2 t \to a = \frac{1}{4c^2 t} \to g(x) = \exp\left(\frac{-x^2}{4c^2 t}\right)$$

$$e^{-c^2 w^2 t} = \frac{1}{c\sqrt{2t}} F(g)$$

$$\hat{u}(w,t) = \frac{1}{c\sqrt{2t}} \hat{f}(w) \hat{g}(w)$$

$$F(f*g) = \sqrt{2\pi} \hat{f}(w) \hat{g}(w)$$

$$\hat{u}(w,t) = \frac{1}{2c\sqrt{\pi t}} F(f*g)$$

$$u(w,t) = \frac{1}{2c\sqrt{\pi t}} f*g$$

$$f * g = \int_{-\infty}^{\infty} f(x) g(x-\tau) d\tau$$

$$u(w,t) = \frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^{\infty} f(x) \exp\left(-\frac{(x-\tau)^2}{4c^2 t}\right) d\tau \qquad \square$$

Oppg. 2a)

$u_t = u_{xx}$, $u(0,t) = u(2,t) = 0$

Antar $u(x,t) = F(x) G(t)$

$G'(t) F(x) = F''(x) G(t)$

$\frac{G'(t)}{G(t)} = \frac{F''(x)}{F(x)} = -k^2 \rightarrow$ $k \in \mathbb{R}/0$!

$F''(x) + k^2 F(x) = 0$

$F(x) = A\sin(kx) + B\cos(kx)$

$F(0) = 0 \rightarrow B = 0$

$F(2) = 0 \rightarrow k = \frac{n\pi}{2}$

$\underline{F(x) = A_n \sin\left(\frac{n\pi}{2} x\right)}$

$G'(t) + k^2 G(t) = 0$

$\underline{G(t) = C_n e^{-k^2 t}}$

$$u(x,0) = \sum_{n=1}^{\infty} A_n e^{-\frac{n^2\pi^2}{4}\cdot 0} \sin\left(\frac{n\pi}{2} x\right) = \sin\left(\frac{\pi x}{2}\right)$$

$\rightarrow A_1 = 1$, $A_n = 0 \ \forall n > 1$

$$\underline{u(x,t) = \exp\left(-\left(\frac{\pi}{2}\right)^2 t\right) \sin\left(\frac{\pi x}{2}\right)}$$

2b) $u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{2} x\right) = \begin{cases} x, & 0 \le x \le 1 \\ 2-x, & 1 \le x \le 2 \end{cases} = F(x)$

$$\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{2} x\right) =$$

$$A_n = \frac{1}{2} \int_{-2}^{2} F(x) \sin\left(\frac{n\pi}{2} x\right) dx = \int_0^2 F(x) \sin\left(\frac{n\pi}{2} x\right) dx$$

$$A_n = \int_0^1 x \sin\left(\frac{n\pi}{2} x\right) dx + \int_1^2 (2-x) \sin\left(\frac{n\pi}{2} x\right) dx$$

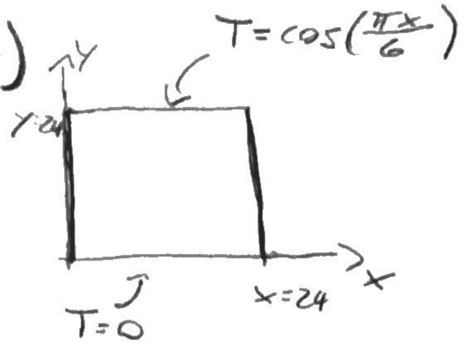
$$A_n = \left[\frac{4}{n^2\pi^2} \sin\left(\frac{n\pi}{2} x\right)\right]_0^1 - \left[\frac{4}{n^2\pi^2} \sin\left(\frac{n\pi}{2} x\right)\right]_1^2 + 2\left[\frac{-2}{n\pi} \cos\left(\frac{n\pi}{2} x\right)\right]_1^2$$

$$A_n = \frac{4}{n^2\pi} \left(\sin\left(\frac{n\pi}{2}\right) - 0 - \left(\sin(n\pi) - \sin\left(\frac{n\pi}{2}\right)\right)\right) - \frac{4}{n\pi}\left(\cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right)$$

$$A_n = \begin{cases} -\frac{4}{n\pi} & , \text{ n er par} \\ \frac{8}{n^2\pi} & , \text{ n er odde} \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{2}x\right) \exp\left(-\left(\frac{n\pi}{2}\right)^2 t\right)$$

Oppg. 4)



$u_{xx} + u_{yy} = 0$

$u(x,0) = 0$

$u(x,24) = \cos\left(\frac{\pi x}{6}\right)$

$u(x,y) = F(x)G(y)$

$F''(x)G(y) + F(x)G''(y) = 0$

$$\frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)} = k^2 \quad , \quad k \in \mathbb{R}/0$$

$G''(y) = -k^2 G(y)$

$\underline{G(y) = A\exp(-ky) + B\exp(ky)}$

$F''(x) = k^2 F(x)$

$\underline{F(x) = C\sin(kx) + D\cos(kx)}$

$u(x,0) = 0 \rightarrow G(0) = 0 \rightarrow \underline{A = -B}$

$u(x,24) = \cos\left(\frac{\pi x}{6}\right) \rightarrow G(24) = 1, \; F(x) = \cos\left(\frac{\pi x}{6}\right) \rightarrow k = \frac{\pi}{6}$

$\quad \hookrightarrow A\exp(-24k) - A\exp(24k) = 1$

$\underline{A = \left(\exp(-4\pi) - \exp(4\pi)\right)^{-1}}$

$$u(x,y) = \frac{\exp\left(-\frac{\pi}{6}y\right) - \exp\left(\frac{\pi}{6}y\right)}{\exp(-4\pi) - \exp(4\pi)} \cos\left(\frac{\pi}{6}x\right) = \frac{\sinh\left(\frac{\pi}{6}y\right)}{\sinh(4\pi)} \cos\left(\frac{\pi}{6}x\right.$$