Matte 4, Øving 6, Vegard G. Jervell

1) $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$, $u(x,0) = F(x)$, $\dfrac{\partial u}{\partial t}\Big|_{(x,0)} = g(x)$

I) $u(x,t) = \varphi(x+ct) + \psi(x-ct)$

II) $u(x,0) = \varphi(x) + \psi(x) = F(x)$

$$\frac{\partial u}{\partial t}\Big|_{(x,0)} = c\varphi'(x) - c\psi'(x) = g(x)$$

$$\varphi'(x) - \psi'(x) = \frac{1}{c} g(x)$$

III) $\varphi(x) - \psi(x) = \dfrac{1}{c}\displaystyle\int_a^x g(\tau)\,d\tau + A$

$$\begin{cases} \text{II} + \text{III}) & 2\varphi(x) = F(x) + \dfrac{1}{c}\displaystyle\int_a^x g(\tau)\,d\tau + A \\ \text{II} - \text{III}) & 2\psi(x) = F(x) - \dfrac{1}{c}\displaystyle\int_a^x g(\tau)\,d\tau - A \end{cases}$$

I) $u(x,t) = \dfrac{1}{2}F(x+ct) + \dfrac{1}{2c}\displaystyle\int_a^{x+ct} g(\tau)\,d\tau + A$

$$+ \frac{1}{2}F(x-ct) - \frac{1}{2c}\int_a^{x-ct} g(\tau)\,d\tau - A$$

$$u(x,t) = \frac{1}{2}\big(F(x+ct) - F(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)\,d\tau$$

Oppg 3)

$$\frac{\partial^2 u}{\partial x^2} = C^2 \frac{\partial^2 u}{\partial t^2} \quad , \quad u(0,t) = u(3,t) = 0$$

$$u(x,t) = F(x)\, G(t)$$

$$F''(x)\, G(t) = C^2 F(x)\, G''(t)$$

$$\frac{F''(x)}{F(x)} = C^2 \frac{G''(t)}{G(t)} = k^2$$

$$F''(x) = k^2 F(x) \wedge k \in \mathbb{R}/0 \to F(x) = A_1 \sin(kx) + B_1 \cos(kx)$$

$$G''(t) = \left(\frac{k}{c}\right)^2 G(t) \wedge \left(\frac{k}{c}\right) \in \mathbb{R}/0 \to G(t) = A_2 \sin\left(\frac{k}{c}t\right) + B_2 \cos\left(\frac{k}{c}t\right)$$

$$u(0,t) = 0 \wedge G(t) \neq 0 \to F(0) = 0 \to \underline{B_1 = 0}$$

$$u(3,t) = 0 \wedge G(t) \neq 0 \to F(3) = 0 \to \sin(3k) = 0 \to \underline{k = \frac{n\pi}{3}}$$

$$u(x,t) = A_1 \sin\left(\frac{n\pi}{3}x\right)\left(A_2 \sin\left(\frac{n\pi}{3c}t\right) + B_2 \cos\left(\frac{n\pi}{3c}t\right)\right)$$

$$\underline{\underline{u(x,t) = \sin\left(\frac{n\pi}{3}x\right)\left(A_n \sin\left(\frac{n\pi}{3c}t\right) + B_n \cos\left(\frac{n\pi}{3c}t\right)\right)}} \quad , \quad C = 1$$

Oppg 4a)

$$u(x,0) = x(3-x), \quad u_t(x,0) = 0$$

$$u_t(x,0) = \sin\left(\frac{n\pi}{3}x\right)\left(A_2 \cos(0) + B_2 \sin(0)\right) = 0 \to \underline{A = 0}$$

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{3}x\right) \cos\left(\frac{n\pi}{3c}t\right)$$

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{3}x\right) = x(3-x)$$

$u(x,0)$ er den odde utvidelsen til $x(3-x)$

$$B_n = \frac{1}{3}\int_{-3}^{3} x(3-x) \sin\left(\frac{n\pi}{3}x\right) dx = \frac{2}{3}\int_0^3 x(3-x) \sin\left(\frac{n\pi}{3}x\right) dx$$

$$B_n = 2\left[\frac{-x}{3}\cos\left(\frac{n\pi}{3}x\right)\right]_0^3 - \frac{2}{3}\left[\frac{2 - a^2x^2}{a^3} \cos ax\right]_0^3$$

$$B_n = -2(-1)^n - \frac{2}{3}\left(\frac{2 - 9a^2}{a^3}(-1)^n - \frac{2}{a^3}\right)$$

$$B_n = 2(-1)^{n-1} + \frac{2}{3}\left(\frac{(n\pi)^2 - 2}{(n\pi)^3} \cdot 27(-1)^n - 2\left(\frac{3}{n\pi}\right)^3\right)$$

4b) $u(x,0) = B_3 \sin\left(\frac{n\pi}{3}x\right) = 0 \rightarrow B_3 = 0$

$$U_t(x,0) = \sum_{n=1}^{\infty} A_n \frac{n\pi}{3} \sin\left(\frac{n\pi}{3}x\right) = \sin \pi x - \frac{1}{2}\sin(2\pi x)$$

$$\rightarrow A_3 = \frac{1}{\pi}, \quad A_6 = \frac{-1}{4\pi}$$

$$u(x,t) = \sin(\pi x)\left(\frac{1}{\pi}\sin(\pi t)\right) - \frac{1}{4\pi}\sin(2\pi x)\sin(2\pi t)$$

4c) $u(x,0) = x(3-x)$

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{3}x\right) = x(3-x)$$

$$B_n = \frac{1}{3}\int_{-3}^{3} x(3-x)\sin\left(\frac{n\pi}{3}x\right)dx$$

$$B_n = 2(-1)^{n-1} + \frac{2}{3}\left(\frac{(n\pi)^2-2}{(n\pi)^3}\cdot 3^3(-1)^n - 2\left(\frac{3}{n\pi}\right)^3\right)$$

$u_t(x,0) = \sin(\pi x) - \frac{1}{2}\sin(2\pi x)$

$$\sum_{n=1}^{\infty} A_n \frac{n\pi}{3}\sin\left(\frac{n\pi}{3}x\right) = U_t(x,0)$$

$$\rightarrow A_3 = \frac{1}{\pi}, \quad A_6 = \frac{-1}{4\pi}$$