Øving 2 Vegard G. Jervell

1) $f*g = \int_0^t f(p)g(t-p)dp$

$$\mathcal{L}(f*g) = \int_0^\infty \int_0^t f(p)g(t-p)dp\, e^{-st}dt$$

$$= \int_0^\infty \int_0^\infty f(p)g(u)e^{-s(u+p)}dp\,du \;, \quad u = t-p \rightarrow du = dt$$

$$= \int_0^\infty f(p)e^{-sp}dp \int_0^\infty g(u)e^{-su}du$$

$$\mathcal{L}(f*g) = \mathcal{L}(f)\mathcal{L}(g) \quad \blacksquare$$

2)
```
import numpy as np, matplotlib.pyplot as plt
plt.plot([-np.pi, 0], [0,0], color='b')
plt.plot([0, np.pi], [1,1], color='b')
plt.show()
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3a) $y'' + 4y' + 5y = \delta(t-1) \;, \quad y(0) = 0 \;, \quad y'(0) = 3$

$$\mathcal{L}(y'') + 4\mathcal{L}(y') + 5\mathcal{L}(y) = \mathcal{L}(\delta(t-1))$$

$$s^2\mathcal{L}(y) - sy(0) - y'(0) + 4(s\mathcal{L}(y) - y(0)) + 5\mathcal{L}(y) = e^{-s}$$

$$\mathcal{L}(y)(s^2+4s+5) = e^{-s} + 3$$

$$\mathcal{L}(y) = \frac{e^{-s}+3}{s^2+4s+5} = \frac{e^{-s}}{s^2+4s+5} + \frac{3}{s^2+4s+5}$$

$$\mathcal{L}(y) = \frac{e^{-s}}{(s+2)^2+1} + \frac{3}{(s+2)^2+1}$$

$$\mathcal{L}(y) = e^{-s}F(s+2) + 3F(s+2) \;, \quad F(u) = \frac{1}{u^2+1}$$

$$\underline{\underline{y = e^{-2t-1}\sin(t-1)u(t-1) + 3e^{-2t}\sin(t)}}$$

36) $y''+5y'+6y=\delta(t-\frac{\pi}{2})+u(t-\pi)\cos(t)$, $y(0)=y'(0)=0$

$$s^2\mathcal{L}(y)+5s\mathcal{L}(y)+6\mathcal{L}(y)=e^{-\frac{\pi}{2}s}+\mathcal{L}(-u(t-\pi)\cos(t-\pi))$$

$$\mathcal{L}(y)(s^2+5s+6)=e^{-\frac{\pi}{2}s}-e^{-\pi s}\left(\frac{s}{s^2+\pi^2}\right)$$

$$\mathcal{L}(y)(s^2+5s+6)=e^{-\frac{\pi}{2}s}\left(1-e^{-\frac{\pi}{2}s}\left(\frac{s}{s^2+\pi^2}\right)\right)$$

$$\mathcal{L}(y)=e^{-\frac{\pi}{2}s}\frac{1}{(s+\frac{5}{2})^2-\frac{1}{4}}-e^{-\pi s}\left(\frac{1}{(s+\frac{5}{2})^2-\frac{1}{4}}\right)\left(\frac{s}{s^2+\pi^2}\right)$$

$$\mathcal{L}^{-1}\left(e^{-\frac{\pi}{2}s}\frac{1}{(s+\frac{5}{2})^2-\frac{1}{4}}\right)=2\mathcal{L}^{-1}\left(e^{-\frac{\pi}{2}s}\frac{\frac{1}{2}}{(s+\frac{5}{2})^2-(\frac{1}{2})^2}\right)=\underline{2\exp(-\frac{5}{2}t-\frac{\pi}{2})\sinh(\frac{t}{2}-\frac{\pi}{2})u(t-\frac{\pi}{2})}$$

$$\mathcal{L}^{-1}\left(e^{-\pi s}\left(\frac{1}{(s+\frac{5}{2})^2-(\frac{1}{2})^2}\right)\left(\frac{s}{s^2+\pi^2}\right)\right)=\int_0^t f(p)g(t-p)\,dp$$

hvor $f=\mathcal{L}^{-1}\left(\frac{1}{(s+\frac{5}{2})^2-(\frac{1}{2})^2}\right)=2e^{-\frac{5}{2}t}\sinh(\frac{t}{2})$

og $g=\mathcal{L}^{-1}\left(e^{-\pi s}\frac{s}{s^2+\pi^2}\right)=\cos(\pi t-\pi)u(t-\pi)$

3c) $ty'' - ty' + y = 1 \quad , \quad y(0) = 1 \; , \; y'(0) = 2$

$$\mathcal{L}(ty'') - \mathcal{L}(ty') + \mathcal{L}(y) = \frac{1}{s}$$

$$-\frac{d}{ds}\left(s^2\mathcal{L}(y) - s - 2\right) + \frac{d}{ds}\left(s\mathcal{L}(y) - 1\right) + \mathcal{L}(y) = \frac{1}{s}$$

$$-2s\mathcal{L}(y) - s^2\frac{d}{ds}\mathcal{L}(y) + 1 + \mathcal{L}(y) + s\frac{d}{ds}\mathcal{L}(y) + \mathcal{L}(y) = \frac{1}{s}$$

$$\mathcal{L}(y)(2 - 2s) + \frac{d}{ds}(\mathcal{L}(y))(s - s^2) = \frac{1}{s} - 1$$

$$2\mathcal{L}(y) + s\frac{d}{ds}\mathcal{L}(y) = \frac{1-s}{s(1-s)}$$

$$2s\mathcal{L}(y) + s^2\frac{d}{ds}\mathcal{L}(y) = 1$$

$$\frac{d}{ds}\left(s^2\mathcal{L}(y)\right) = 1$$

$$\mathcal{L}(y) = \frac{s + C}{s^2} = \frac{1}{s} + \frac{C}{s^2}$$

$$\underline{\underline{y = 1 + Ct}}$$

3d) $y(t) - \int_0^t y(\tau)(t-\tau)\,d\tau = 2 - \frac{1}{2}t^2$

$$\mathcal{L}(y) - \mathcal{L}(y * t) = \frac{2}{s} - \frac{1}{2}\left(\frac{2}{s^3}\right)$$

$$\mathcal{L}(y) - \frac{1}{s^2}\mathcal{L}(y) = \frac{2}{s} - \frac{1}{s^3}$$

$$\mathcal{L}(y) = \left(\frac{2s^2-1}{s^3}\right)\left(\frac{1}{1-\frac{1}{s^2}}\right)$$

$$\mathcal{L}(y) = \frac{(2s^2-1)s^2}{s^3(s^2-1)}$$

$$\mathcal{L}(y) = \frac{1}{s}\left(\frac{s^2}{s^2-1} + 1\right)$$

$$\mathcal{L}(y) = \frac{s}{s^2-1} + \frac{1}{s}$$

$$\underline{\underline{y = \cosh(t) + 1}}$$