TMA4115 Calculus 3
Spring 2018

Norwegian University of Science and Technology
Department of Mathematical Sciences

Work Sheet Week 15

Remark: Hand in the exercises C.1–C.3.

A — Reading

PBA = Polking–Boggess–Arnold (the part of the book that deals with differential equations)

PBA 4.5 Inhomogeneous Equations; the Method of Undetermined Coefficients
PBA 4.6 Variation of Parameters
PBA 4.2 (Second-Order Equations and) Systems

B — Finger Exercises

B.1

Find a solution to each of these equations using undetermined coefficients.

$y''(t) - y'(t) - 2y(t) = 2\mathrm{e}^{-2t}$

$y''(t) - y'(t) - 2y(t) = 3\mathrm{e}^{-t}$

$y''(t) + y'(t) + y(t) = t$

$y''(t) + y'(t) + 2y(t) = 2\cos(2t)$

$y''(t) - 9y(t) = \mathrm{e}^{3t}$

B.2

Find a solution to the equation

$$y''(t) + 2y'(t) - 3y(t) = \mathrm{e}^{-t}$$

using variation of parameters, and also using undetermined coefficients. Compare the results.

B.3

Rewrite the forth order differential equation

$$y^{(4)}(t) - y^{(2)}(t) - 2y(t) = 0$$

as a system of four first order differential equations: $\mathbf{v}'(t) = A\mathbf{v}(t)$. Compute the characteristic polynomial of the matrix A, and compare it with the characteristic polynomial of the differential equation.

C — Exam Preparation

The goal of the following exercises is to solve the linear system of differential equations

$$\begin{aligned} x'(t) &= x(t) + y(t) \\ y'(t) &= 3x(t) - y(t) \end{aligned}$$

with initial conditions $x(0) = 1$ and $y(0) = 2$. The system can be written $\mathbf{v}'(t) = A\mathbf{v}(t)$ with the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$$

and the (time-dependent) vector

$$\mathbf{v}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

If $\mathbf{v} = v$ were scalar, the solution of the equation $v'(t) = av(t)$ with initial condition $v(0) = v_0$ is $v(t) = \mathrm{e}^{at}v_0$. We will therefore try to make sense of the expression $\mathbf{v}(t) = \mathrm{e}^{At}\mathbf{v}_0$ in order to find a solution of the system above. The formula that we can use is the Taylor expansion

$$\mathrm{e}^{tX} = \mathrm{I} + tX + \frac{t^2}{2}X^2 + \cdots + \frac{t^n}{n!}X^n + \cdots \tag{1}$$

for the exponential. (The I denotes the identity matrix.)

C.1 Find a formula for all A^n where A is the matrix above. Hint: use diagonalization.

C.2 Calculate e^{At}. Hint: use C.1 and the Taylor expansion (1).

C.3 Check that $\mathbf{v}(t) = \mathrm{e}^{At}\mathbf{v}_0$ with

$$\mathbf{v}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

solves the system with the given initial conditions.