Norwegian University of Science and Technology
Department of Mathematical Sciences

Work Sheet Week 6

Remark: Hand in **C.1** and 2 of 3 exercises (of your own choice) from **C.2-C.4**.

A - Reading

L = Lay (the part of the textbook that deals with linear algebra)

L 1.7 Linear Independence

L 1.8 Introduction to Linear Transformations

L 1.9 Matrices of Linear Transformations

B - Finger Exercises

B.1

Consider

$$\begin{bmatrix}1\\0\\2\\0\end{bmatrix}, \quad \begin{bmatrix}0\\0\\1\\-1\end{bmatrix}, \quad \begin{bmatrix}4\\-3\\-2\\4\end{bmatrix}, \quad \begin{bmatrix}3\\-2\\1\\1\end{bmatrix}$$

in $\mathbb{R}^4$.

Are the vectors linearly independent? Do they span $\mathbb{R}^4$?

B.2

Consider

$$\mathbf{u} = \begin{bmatrix}1\\2\\3\end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix}4\\5\\6\end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix}7\\8\\9\end{bmatrix}$$

in $\mathbb{R}^3$.

Write $\mathbf{u}$ as a linear combination of $\mathbf{v}$ and $\mathbf{w}$, $\mathbf{v}$ as a linear combination of $\mathbf{u}$ and $\mathbf{w}$, and $\mathbf{w}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

Are the three vectors linearly independent? If so, do they span $\mathbb{R}^3$? If not, describe their span by means of a linear equation.

B.3

Consider the linear transformation $T : \mathbb{C}^4 \to \mathbb{C}^3$ given by

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 - x_3 + 3x_4 \\ -2x_1 + 4x_2 + 5x_3 - 5x_4 \\ 3x_1 - 6x_2 - 6x_3 + 8x_4 \end{bmatrix}$$

Find the matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{C}^4$. Is T onto? Is it one-to-one? Explain.

Do the same for $S : \mathbb{C}^3 \to \mathbb{C}^3$ given by

$$S \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \lambda x_2 \\ \lambda x_3 \\ \lambda x_1 \end{bmatrix},$$

where λ is a given complex number.

B.4

Let

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 0 \end{bmatrix}.$$

Find a matrix that describes T. Explain whether or not T is onto or one-to-one.

B.5

Let

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Describe the associated linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(\mathbf{x}) = H\mathbf{x}$, in geometric terms.

Is T onto? Is it one-to-one? Explain.

C - Exam Preparation

C.1

Complete the following definitions:

A set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ of vectors in $\mathbb{C}^n$ is *linearly independent* if and only if...

The *span* of a set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ of vectors in $\mathbb{C}^n$ is...

A transformation $T : \mathbb{C}^m \to \mathbb{C}^n$ is *linear* if and only if...

A transformation $T : \mathbb{C}^m \to \mathbb{C}^n$ is *onto* if and only if...

A transformation $T : \mathbb{C}^m \to \mathbb{C}^n$ is *one-to-one* if and only if...

C.2

Let $T : \mathbb{C}^m \to \mathbb{C}^n$ be a linear transformation. For each of the three cases $m < n$, $m > n$ and $m = n$, mark the following statements *true* or *false*.

T cannot be one-to-one.

T cannot be onto.

C.3

Give examples of functions $f : \mathbb{R} \to \mathbb{R}$ that are

1. onto and one-to-one,
2. onto but not one-to-one,
3. not onto but one-to-one,
4. neither onto nor one-to-one.

C.4

Let A be a matrix and $\mathbf{v}$ and $\mathbf{w}$ non-zero vectors with $A\mathbf{v} = \mathbf{v}$ and $A\mathbf{w} = -\mathbf{w}$. Show that $\mathbf{v}$ and $\mathbf{w}$ are linearly independent.